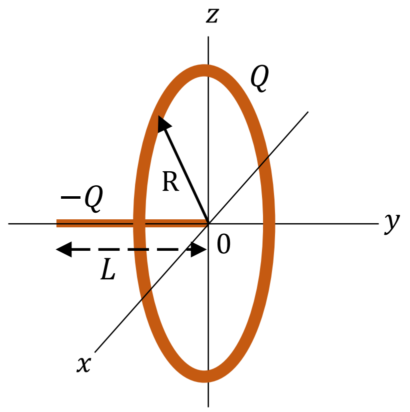
PHYS 102 – General Physics II Midterm Exam

Duration: 120 minutes Saturday, 22 October 2022; 13:30

1. A charge distribution consists of a ring with radius R and total charge Q (uniformly distributed) placed on the xz plane such that its center is at the origin, and a line of length L and total charge $-Q$ (uniformly distributed) placed on the y axis where $-L < y < 0$.



(a) (15 Pts.) Find the electric field on the y axis for $0 < y < \infty$.

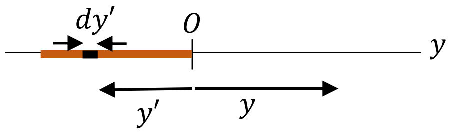
(b) (10 Pts.) Find the force on the line charge.

Solution:

(a) Since the charge distribution is symmetric, electric field on the y axis will be in the y direction. Any point on the y axis is at a distance $\sqrt{y^2+R^2}$ to all points on the ring. Therefore, the electric field due to the ring is is found as

$$|E_{y\text{ ring}}| = \frac{1}{4\pi\epsilon_0}\int\frac{y\,dq}{(y^2+R^2)^{3/2}} = \frac{y}{4\pi\epsilon_0(y^2+R^2)^{3/2}}\int dq = \frac{Q}{4\pi\epsilon_0}\frac{y}{(y^2+R^2)^{3/2}}.$$

Magnitude of the electric field due to the line charge is found as



$$|E_{y\text{ line}}| = \frac{Q/L}{4\pi\epsilon_0}\int_0^L\frac{dy'}{(y+y')^2} = \frac{Q}{4\pi\epsilon_0 L}\left(\frac{1}{y}-\frac{1}{y+L}\right) = \frac{Q}{4\pi\epsilon_0 y(y+L)}.$$

$$\vec{\mathbf{E}} = \frac{Q}{4\pi\epsilon_0}\left[\frac{y}{(y^2+R^2)^{3/2}} - \frac{1}{y(y+L)}\right]\hat{\mathbf{j}}.$$

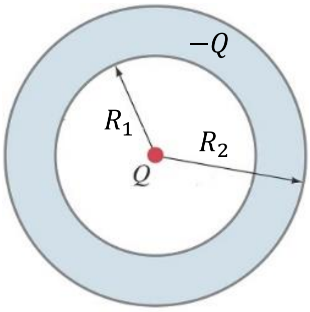
(b) Magnitude of the force on the infinitesimal piece is $dF = E_{y\text{ ring}}\,dq$, where $dq = (Q/L)dy'$.

$$\mathrm{dF} = \frac{Q}{4\pi\epsilon_0}\frac{y'}{(y'^2+R^2)^{3/2}}\frac{Q}{L}dy' \quad\rightarrow\quad \mathrm{F} = \frac{Q^2}{4\pi\epsilon_0 L}\int_0^L\frac{y'dy'}{\left(y'^2+R^2\right)^{3/2}}.$$

The integral can be evaluated using the substitution $u^2 = y'^2 + R^2 \quad\rightarrow\quad y'dy' = u\,du$. Since charge on the line is negative while charge on the ring is positive, the force is attractive.

$$\vec{\mathbf{F}} = \frac{Q^2}{4\pi\epsilon_0 L}\left[\frac{1}{R} - \frac{1}{\sqrt{L^2+R^2}}\right]\hat{\mathbf{j}}.$$

2. A conducting spherical shell with inner radius R_1, and outer radius R_2 has a point charge Q at its center. A charge $-Q$ is put on the conductor.



(a) (9 Pts.) Find the magnitude of the electric field both inside and outside the shell.

(b) (8 Pts.) Find the surface charge density on the inner and the outer surfaces of the shell.

(c) (8 Pts.) Find the electric potential both inside and outside the shell taking $\lim_{r\to\infty} V(r) = 0$.

Solution:

(a) Consider a spherical Gaussian surface of radius r concentric with the conducting shell. For $0 < r < R_1$, charge enclosed by the Gaussian surface is Q. Therefore, Gauss's law implies

$$\oint \vec{\mathbf{E}} \cdot d\vec{\boldsymbol{A}} = E(r) \oint dA = E(r)(4\pi r^2) = \frac{Q}{\epsilon_0} \quad \rightarrow \quad E(r) = \frac{Q}{4\pi\epsilon_0 r^2}\,, \qquad 0 < r < R_1\,.$$

For $R_2 < r < \infty$, net charge enclosed by the Gaussian surface is 0. Therefore, $E(r) = 0\,,\ R_2 \le r < \infty\,.$

(b) Since electric field inside the conductor is zero, net charge enclosed by a Gaussian surface with $R_1 < r \le R_2$ must be zero. Therefore, the total charge induced on the inner surface of the shell must be $-Q$. This means the surface charge density on the inner surface is

$$\sigma_1 = \frac{-Q}{A} = \frac{-Q}{4\pi R_1^2}\,, \qquad \sigma_2 = 0\,.$$

Similarly, since $E(r) = 0$ when $R_2 < r < \infty$, we have $\sigma_2 = 0$ on the outer surface of the shell.

(c) $E(r) = 0$ when $R_2 < r < \infty$ implies $V(r)$ is constant in that region. If $\lim_{r\to\infty} V(r) = 0$, the constant must be zero. Similarly, electric feld inside the conductor is zero. Therefore, we have

$$V = 0\,, \qquad R_1 < r < \infty\,.$$

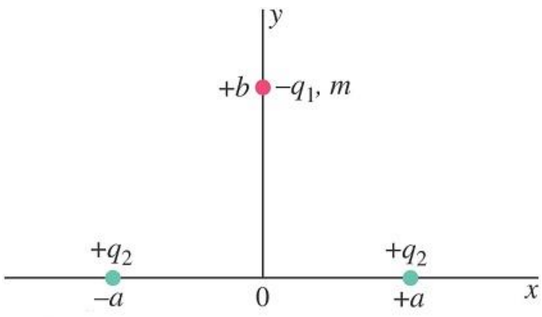
For , $0 < r < R_1$, we have

$$V(r) = -\int_{\infty}^{r} E(r')dr' = -\int_{R_1}^{r} E(r')dr' = -\frac{Q}{4\pi\epsilon_0}\int_{R_1}^{r}\frac{dr'}{r'^2} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{R_1}\right), \qquad 0 < r < R_1\,.$$

3. A charge $-q_1$ of mass *m* rests on the *y* axis at a distance *b* above the *x* axis. Two positive charges of magnitude q_2 are fixed on the *x* axis at $x = -a$ and $x = a$ respectively.

(a) (13 Pts.) Find the potential energy (self-energy) of this charge distribution.

(b) (12 Pts.) If the particle with charge $-q_1$ is given an initial velocity v_0 in the positive *y* direction, what is the minimum value of v_0 such that the charge escapes to a point infinitely far away from the two positive charges?



Solution:

(a)

$$U = \frac{q_2^2}{8\pi\epsilon_0 a} - \frac{q_1 q_2}{4\pi\epsilon_0\sqrt{a^2+b^2}} - \frac{q_1 q_2}{4\pi\epsilon_0\sqrt{a^2+b^2}} = \frac{q_2}{2\pi\epsilon_0}\left(\frac{q_2}{4a} - \frac{q_1}{\sqrt{a^2+b^2}}\right).$$

(b) Initial total mechanical energy of the particle with charge $-q_1$ is

$$E_i = \frac{1}{2}mv_0^2 - \frac{q_1 q_2}{2\pi\epsilon_0\sqrt{a^2+b^2}}.$$

If the charge escapes to infinity, its potential energy will be zero. For minimum value of v_0, we must have $E_f = 0$. Since total mechanical energy is conserved,

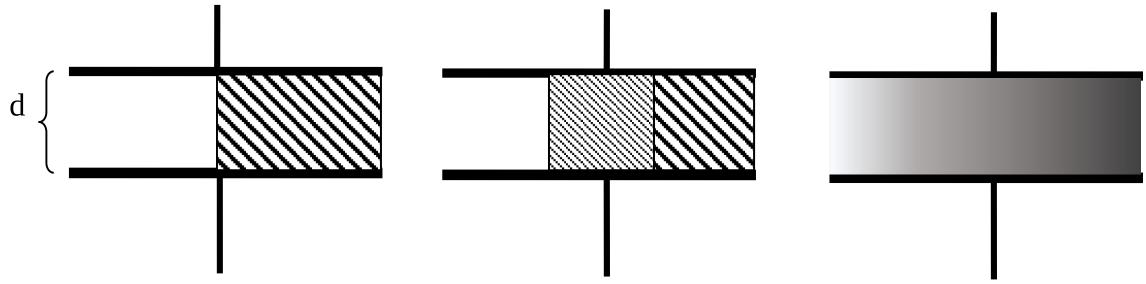
$$E_i = E_f = 0 \quad \rightarrow \quad v_{0\,\mathrm{min}} = \sqrt{\frac{q_1 q_2}{\pi m\epsilon_0\sqrt{a^2+b^2}}}.$$

4. Consider capacitors made from square metal plates of side L, which are parallel with a distance d between them. Calculate the capacitance of the system for the following arrangements of dielectric material between the plates.

(a) (8 Pts.) Half of the space is filled with material of dielectric constant κ as shown in figure on the left.

(b) (8 Pts.) One third of the space is filled with material of dielectric constant of $(1+\kappa)/2$, another one third with κ as shown in the middle figure.

(c) (9 Pts.) The dielectric function of the material filling the space is linearly increasing from 1 at the left boundary to κ at the right boundary, i.e., $\kappa(x) = \kappa + \frac{L-x}{L}(1-\kappa)$, as shown in figure on the right.



Solution: In the first two cases, systems can be treated as connected capacitors, some with a dielectric, and one without a dielectric. All capacitors have their high voltage plates in contact and their low voltage plates in contact, so they are in parallel. In the last case, the dielectric function varies continuously, so we consider infinitesimally sliced capacitors connected in parallel.

(a)

$$C_{\text{eq}} = C_1 + C_2 = \epsilon_0 \frac{L(L/2)}{d} + \kappa\epsilon_0 \frac{L(L/2)}{d} = \left(\frac{1+\kappa}{2}\right)\epsilon_0 \frac{L^2}{d}.$$

(b)

$$C_{\text{eq}} = C_1 + C_2 + C_3 = \epsilon_0 \frac{L(L/3)}{d} + \left(\frac{1+\kappa}{2}\right)\epsilon_0 \frac{L(L/3)}{d} + \kappa\epsilon_0 \frac{L(L/3)}{d},$$

$$C_{\text{eq}} = \epsilon_0 \frac{L^2}{3d}\left(1 + \frac{1+\kappa}{2} + \kappa\right) = \left(\frac{1+\kappa}{2}\right)\epsilon_0 \frac{L^2}{d}.$$

(c)

$$C_{\text{eq}} = \int_0^L \kappa(x)\epsilon_0 \frac{L}{d} dx = \epsilon_0 \frac{L}{d}\int_0^L \left[\kappa + \frac{L-x}{L}(1-\kappa)\right] dx\,,$$

$$C_{\text{eq}} = \epsilon_0 \frac{L}{d}\left[\kappa \int_0^L dx + (1-\kappa)\int_0^L \left(1 - \frac{x}{L}\right) dx\right] = \epsilon_0 \frac{L}{d}\left[\kappa L + (1-\kappa)\frac{L}{2}\right],$$

$$C_{\text{eq}} = \left(\frac{1+\kappa}{2}\right)\epsilon_0 \frac{L^2}{d}.$$